When are wreath products Hopfian?

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Proof.

...something something HNN extension...something something Britten's Lemma. □

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Exercise

The group $A_5 \wr \mathbb{Z}$ *is Hopfian.*

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Δ *non-Hopfian* $\Rightarrow \Delta \wr \Gamma$ *non-Hopfian.*

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